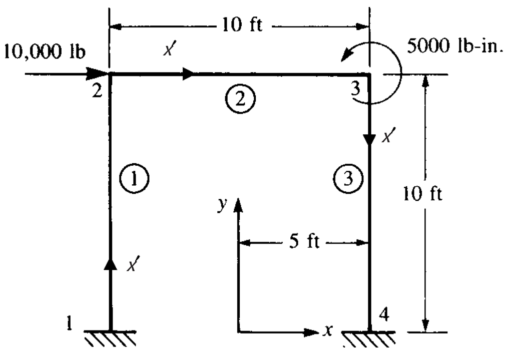

Figure 5–4 Plane frame for analysis, also showing local x' axis for each element

at node 3. The global-coordinate axes and the element lengths are shown in Figure 5–4.

Let $E = 30 \times 10^6$ psi and $A = 10$ in.^{2} for all elements, and let $I = 200$ in.^{4} for elements 1 and 3, and $I = 100$ in.^{4} for element 2.

Using Eq. (5.1.11), we obtain the global stiffness matrices for each element.

Element 1

For element 1, the angle between the global x and the local x' axes is 90 (counterclockwise) because x' is assumed to be directed from node 1 to node 2. Therefore,

$$C = \cos 90 = \frac{x_2 - x_1}{L^{(1)}} = \frac{-60 - (-60)}{120} = 0$$

$$S = \sin 90 = \frac{y_2 - y_1}{L^{(1)}} = \frac{120 - 0}{120} = 1$$

Also,

$$\frac{12I}{L^2} = \frac{12(200)}{(10 \times 12)^2} = 0.167 \text{ in.}^2 \tag{5.2.1}$$

$$\frac{6I}{L} = \frac{6(200)}{10 \times 12} = 10.0 \text{ in.}^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{10 \times 12} = 250{,}000 \text{ lb/in.}^3$$

Then, using Eqs. (5.2.1) to help in evaluating Eq. (5.1.11) for element 1, we obtain the element global stiffness matrix as

$$[k^{(1)}] = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \tag{5.2.2}$$

where all diagonal terms are positive.

Element 2

For element 2, the angle between x and x' is zero because x' is directed from node 2 to node 3. Therefore,

$$C = 1 \qquad S = 0$$

Also,

$$\frac{12I}{L^2} = \frac{12(100)}{120^2} = 0.0835 \text{ in.}^2$$

$$\frac{6I}{L} = \frac{6(100)}{120} = 5.0 \text{ in.}^3 \tag{5.2.3}$$

$$\frac{E}{L} = 250{,}000 \text{ lb/in.}^3$$

Using the quantities obtained in Eqs. (5.2.3) in evaluating Eq. (5.1.11) for element 2, we obtain

$$[k^{(2)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & \phi_2 & u_3 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & -0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \tag{5.2.4}$$

Element 3

For element 3, the angle between x and x' is 270 (or -90) because x' is directed from node 3 to node 4. Therefore,

$$C = 0 \qquad S = -1$$

Therefore, evaluating Eq. (5.1.11) for element 3, we obtain

$$[k^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & \phi_3 & u_4 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \tag{5.2.5}$$

Superposition of Eqs. (5.2.2), (5.2.4), and (5.2.5) and application of the boundary conditions $u_1 = v_1 = \phi_1 = 0$ and $u_4 = v_4 = \phi_4 = 0$ at nodes 1 and 4 yield the reduced set of equations for a longhand solution as

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \end{Bmatrix} = 250{,}000 \begin{bmatrix} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} \tag{5.2.6}$$

Solving Eq. (5.2.6) for the displacements and rotations, we have

$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211 \text{ in.} \\ 0.00148 \text{ in.} \\ -0.00153 \text{ rad} \\ 0.209 \text{ in.} \\ -0.00148 \text{ in.} \\ -0.00149 \text{ rad} \end{Bmatrix} \tag{5.2.7}$$

The results indicate that the top of the frame moves to the right with negligible vertical displacement and small rotations of elements at nodes 2 and 3.

The element forces can now be obtained using $\{f'\} = [k'][T]\{d\}$ for each element, as was previously done in solving truss and beam problems. We will illustrate this procedure only for element 1. For element 1, on using Eq. (5.1.10) for $[T]$ and Eq. (5.2.7) for the displacements at node 2, we have

$$[T]\{d\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ u_2 = 0.211 \\ v_2 = 0.00148 \\ \phi_2 = -0.00153 \end{Bmatrix} \tag{5.2.8}$$

On multiplying the matrices in Eq. (5.2.8), we obtain

$$[T]\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix} \tag{5.2.9}$$

Then using $[k'$ from Eq. (5.1.8), we obtain element 1 local forces as

$$\{f'\} = [k'\,[T\,\{d\} = 250{,}000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix} \tag{5.2.10}$$

Simplifying Eq. (5.2.10), we obtain the local forces acting on element 1 as

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} -3700 \text{ lb} \\ 4990 \text{ lb} \\ 376{,}000 \text{ lb-in.} \\ 3700 \text{ lb} \\ -4990 \text{ lb} \\ 223{,}000 \text{ lb-in.} \end{Bmatrix} \tag{5.2.11}$$

A free-body diagram of each element is shown in Figure 5–5 along with equilibrium verification. In Figure 5–5, the x' axis is directed from node 1 to node 2—consistent with the order of the nodal degrees of freedom used in developing the stiffness matrix for the element. Since the x-y plane was initially established as shown in Figure 5–4, the z axis is directed outward—consequently, so is the z' axis (recall $z' = z$). The y' axis is then established such that x' cross y' yields the direction of z'. The signs on the resulting element forces in Eq. (5.2.11) are thus consistently shown in Figure 5–5. The forces in elements 2 and 3 can be obtained in a manner similar to that used to obtain Eq. (5.2.11) for the nodal forces in element 1. Here we report only the final results for the forces in elements 2 and 3 and leave it to your discretion to perform the detailed calculations. The element forces (shown in Figures 5–5(b) and (c)) are as follows:

Element 2

$$\begin{aligned} f'_{2x} &= 5010 \text{ lb} \qquad & f'_{2y} &= -3700 \text{ lb} \qquad & m'_2 &= -223{,}000 \text{ lb-in.} \\ f'_{3x} &= -5010 \text{ lb} \qquad & f'_{3y} &= 3700 \text{ lb} \qquad & m'_3 &= -221{,}000 \text{ lb-in.} \end{aligned} \tag{5.2.12a}$$

Element 3

$$\begin{aligned} f'_{3x} &= 3700 \text{ lb} \qquad & f'_{3y} &= 5010 \text{ lb} \qquad & m'_3 &= 226{,}000 \text{ lb-in.} \\ f'_{4x} &= -3700 \text{ lb} \qquad & f'_{4y} &= -5010 \text{ lb} \qquad & m'_4 &= 375{,}000 \text{ lb-in.} \end{aligned} \tag{5.2.12b}$$

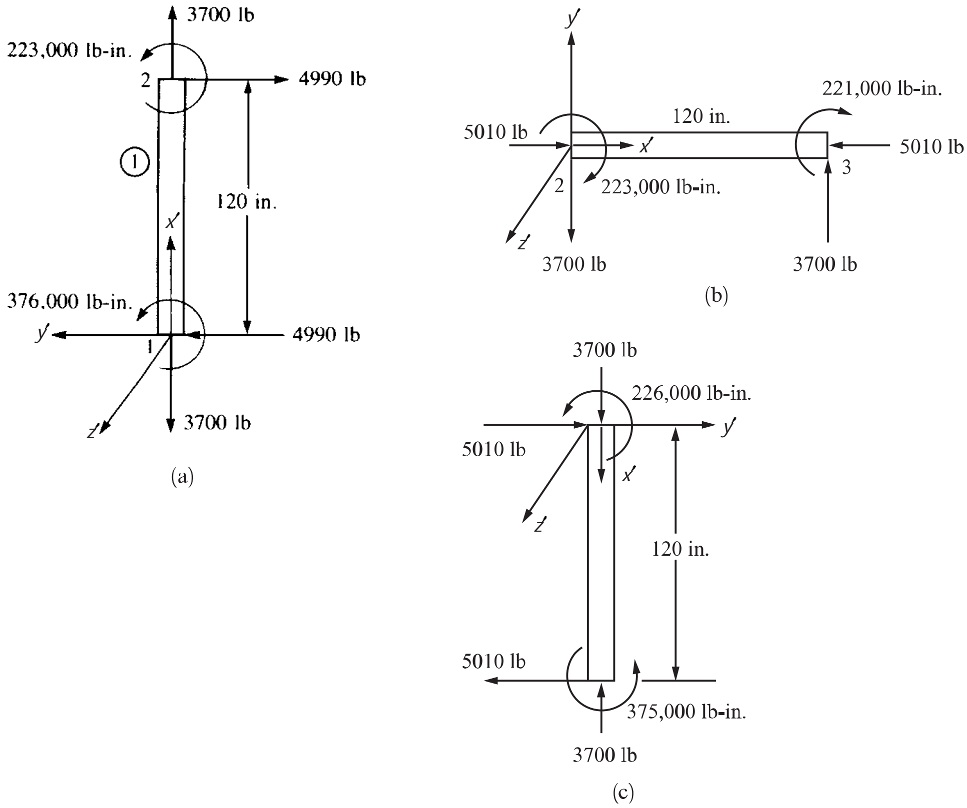

Figure 5–5 Free-body diagrams of (a) element 1, (b) element 2, and (c) element 3

Considering the free body of element 1, the equilibrium equations are

$$\sum F_{x'}:\ -4990 + 4990 = 0$$

$$\sum F_{y'}:\ -3700 + 3700 = 0$$

$$\sum M_2:\ 376{,}000 + 223{,}000 - 4990(120 \text{ in.}) \cong 0$$

Considering moment equilibrium at node 2, we see from Eqs. (5.2.12a) and (5.2.12b) that on element 1, $m_2' = 223{,}000$ lb-in., and the opposite value, $-223{,}000$ lb-in., occurs on element 2. Similarly, moment equilibrium is satisfied at node 3, as m_3' from elements 2 and 3 add to the 5000 lb-in. applied moment. That is, from Eqs. (5.2.12a) and (5.2.12b) we have

$$-221{,}000 + 226{,}000 = 5000 \text{ lb-in.} \qquad \blacksquare$$