2.2.3 Electromagnetic waves at a boundary and Fresnel's equations

We have studied wave propagation through an unbounded medium so far. In this section, we discuss wave propagation through a boundary between two semi-infinite media sharing a common interface. Specifically, we first investigate the effects of wave polarization upon reflection and transmission at the interface between two linear, isotropic and homogeneous media and derive the *Fresnel's equations*. We then include a discussion on *total internal reflection* and establish the properties of *evanescent waves.*

We consider a plane polarized wave incident on the interface at an angle θ_i with respect to the normal of the interface as shown in Fig. 2.2. The plane containing the incident propagation vector $\boldsymbol{k}_i$ and the normal to the interface is called the *plane of incidence*. Since a vector field lying on a plane in an arbitrary direction always can be decomposed into two orthogonal directions. We choose to decompose the $\boldsymbol{\mathcal{E}}$ field into a direction perpendicular and the other parallel to the plane of incidence. We consider these two cases separately, and the general situation is obtained by superposing the results of the two cases.

Parallel polarization

With reference to Fig. 2.2, we take the fields of the incident, reflected, and transmitted waves to be of the following forms, respectively

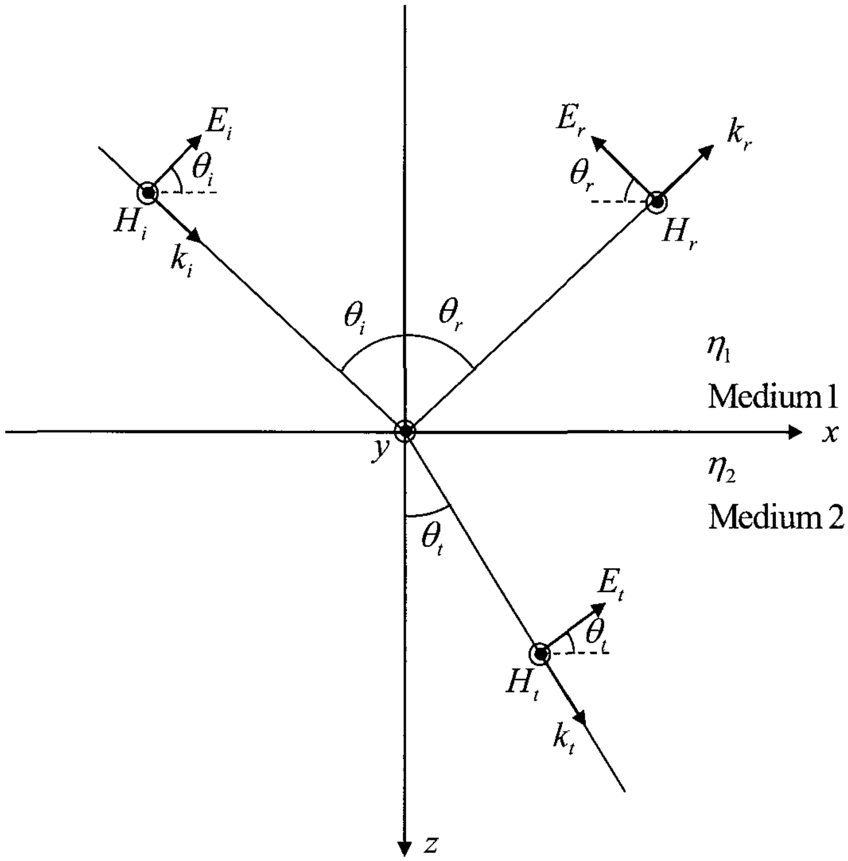


Fig. 2.2 Parallel polarization.

$$\boldsymbol{E}_i = \boldsymbol{E}_{i0}\exp[-j\,(\boldsymbol{k}_i \cdot \boldsymbol{R})] = \boldsymbol{E}_{i0}\exp[-j(k_i\sin\theta_i x + k_i\cos\theta_i z)],$$

$$\boldsymbol{E}_r = \boldsymbol{E}_{r0}\exp[-j\,(\boldsymbol{k}_r \cdot \boldsymbol{R})] = \boldsymbol{E}_{r0}\exp[-j(k_r\sin\theta_r x - k_r\cos\theta_r z)],$$

and

$$\boldsymbol{E}_t = \boldsymbol{E}_{i0}\exp[-j\,(\boldsymbol{k}_t \cdot \boldsymbol{R})] = \boldsymbol{E}_{t0}\exp[-j(k_t\sin\theta_t x + k_t\cos\theta_t z)]. \tag{2.2-41a}$$

Similarly, for the magnetic fields, we have

$$\boldsymbol{H}_i = \boldsymbol{H}_{i0}\exp[-j\,(\boldsymbol{k}_i \cdot \boldsymbol{R})] = \boldsymbol{H}_{i0}\exp[-j(k_i\sin\theta_i x + k_i\cos\theta_i z)],$$

$$\boldsymbol{H}_r = \boldsymbol{H}_{r0}\exp[-j\,(\boldsymbol{k}_r\cdot\boldsymbol{R})] = \boldsymbol{H}_{r0}\exp[-j(k_r\sin\theta_r x - k_r\cos\theta_r z)];$$
$$\boldsymbol{H}_t = \boldsymbol{H}_{t0}\exp[-j\,(\boldsymbol{k}_t\cdot\boldsymbol{R})] = \boldsymbol{H}_{t0}\exp[-j(k_t\sin\theta_t x + k_t\cos\theta_t z)]. \tag{2.2-41b}$$

Note that these electric fields are on the plane of incidence and hence their polarizations are parallel to the plane of incidence. According to the electromagnetic boundary conditions at the interface between two linear, isotropic and homogeneous media with no surface charges and surface currents at the interface, the tangential components of $\mathcal{E}$ and $\mathcal{H}$, and the normal components of $\mathcal{D}$ and $\mathcal{B}$ are continuous across an interface. Continuity of the tangential components of the electric fields at the interface ($z=0$) requires that

$$(\boldsymbol{E}_i + \boldsymbol{E}_r)|_{\,along\ interface} = \boldsymbol{E}_t|_{along\ interface}\,, \tag{2-2-42a}$$

which implies (with reference to Fig. 2.2)

$$E_{i0}\cos\theta_i\exp[-j(k_i\sin\theta_i x)] - E_{r0}\cos\theta_r\exp[-j(k_r\sin\theta_r x)] =$$
$$E_{t0}\cos\theta_t\exp[-j(k_t\sin\theta_t x)], \tag{2.2-42b}$$

where, according to Fig. 2.2, we have

$$\boldsymbol{E}_i = E_{i0}(\cos\theta_i\mathbf{a}_x - \sin\theta_i\mathbf{a}_z),$$
$$\boldsymbol{E}_r = E_{r0}(-\cos\theta_r\mathbf{a}_x - \sin\theta_r\mathbf{a}_z),$$

and

$$\boldsymbol{E}_t = E_{t0}(\cos\theta_t\mathbf{a}_x - \sin\theta_t\mathbf{a}_z).$$

Now, the boundary condition for the tangential component of the magnetic field gives

$$(\boldsymbol{H}_i + \boldsymbol{H}_r)|_{along\ interface} = \boldsymbol{H}_t|_{along\ interface}\,, \tag{2.2-43a}$$

which is equivalent to

$$H_{i0}\exp[-j(k_i\sin\theta_i x)] + H_{r0}\exp[-j(k_r\sin\theta_r x)] =$$
$$H_{t0}\exp[-j(k_t\sin\theta_t x)], \tag{2.2-43b}$$

where $\boldsymbol{H}_i = H_{i0}\mathbf{a}_y$, $\boldsymbol{H}_r = H_{r0}\mathbf{a}_y$, and $\boldsymbol{H}_t = H_{t0}\mathbf{a}_y$. Now, in order to satisfy Eqs. (2.2-42b) and (2.2-43b) for all possible values of x along the interface, all three exponential arguments must be equal and that gives the so-called *phase matching condition:*

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t. \tag{2.2-44}$$

Note that the first equality in the above equation leads to law of reflection and the second equality leads to Snell's law. In light of Eq. (2.2-44), the boundary conditions given by Eqs. (2.2-42b) and (2.2-43b) reduce to

$$E_{i0}\cos\theta_i - E_{r0}\cos\theta_r = E_{t0}\cos\theta_t, \tag{2.2-45a}$$

and

$$H_{i0} + H_{r0} = H_{t0}, \tag{2.2-45b}$$

respectively. Using $H_{i0} = E_{i0}/\eta_1$, $H_{r0} = E_{r0}/\eta_1$ and $H_{t0} = E_{t0}/\eta_2$, where $\eta_1$and η_2 are the intrinsic impedances for medium 1 and 2, Eqs. (2.2-45a) and (2.2-45b) can be solved simultaneously to obtain the *amplitude reflection and transmission coefficients*, $r_{\|}$ and $r_{\perp}$, respectively:

$$r_{\|} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}, \tag{2.2-46a}$$

$$t_{\|} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}. \tag{2.2-46b}$$

Perpendicular polarization

In this case, the electric field vectors are perpendicular to the plane of incidence, as shown in Fig. 2.3. As we did previously in the parallell-polarization case, we can obtain the following expressions for the amplitude reflection and transmission coefficients:

$$r_{\perp} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}, \tag{2.2-47a}$$

$$t_{\perp} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}. \tag{2.2-47b}$$

Equations (2.2-46) and (2.2-27) are called the *Fresnel's equations*, which dictate plane wave reflection and transmission at the interface between two semi-infinite media characterized by η_1 and η_2.

Brewster angle

The incident angle for which the reflection coefficient is zero is called the *Brewster angle* θ_p, also called the *polarizing angle*. For perpendicular polarization, we set $r_\perp = 0$ to get

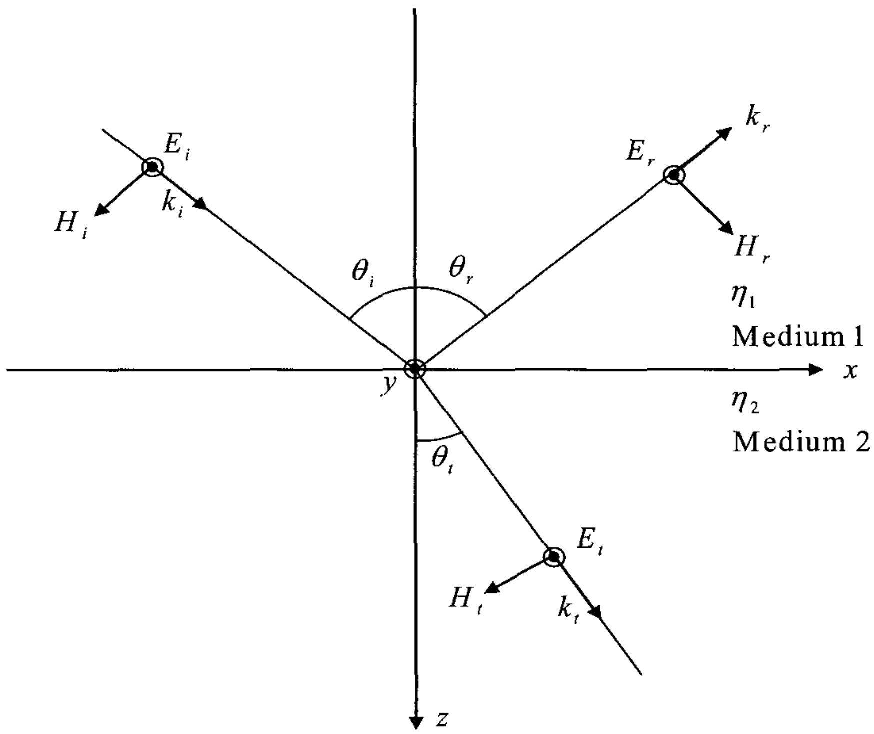


Fig. 2.3 Perpendicular polarization.

$$\eta_2 \cos\theta_i = \eta_1 \cos\theta_t. \tag{2.2-48}$$

Using Eqs. (2.2-12) and (2.2-44), we solve for θ_i in Eq. (2.2-48) to obtain

$$\sin\theta_i = \sqrt{\frac{1 - (\mu_1\epsilon_2/\mu_2\epsilon_1)}{1 - (\mu_1/\mu_2)^2}} = \sin\theta_{p\perp}\,. \tag{2.2-49}$$

When $\mu_1 = \mu_2$, the denominator of Eq. (2.2-48) goes to zero. It means that θ_p does not exist for nonmagnetic materials. Similarly, we find the

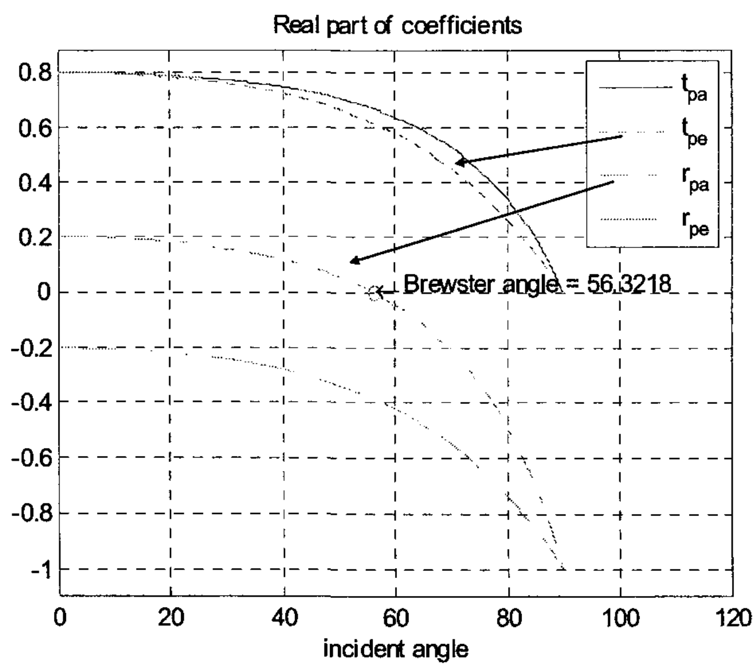
Brewster angle for parallel polarization by setting $r_{\parallel} = 0$ to obtain

$$\sin\theta_{p\parallel} = \sqrt{\frac{1-(\mu_2\epsilon_1/\mu_1\epsilon_2)}{1-(\epsilon_1/\epsilon_2)^2}}. \qquad (2.2\text{-}50)$$

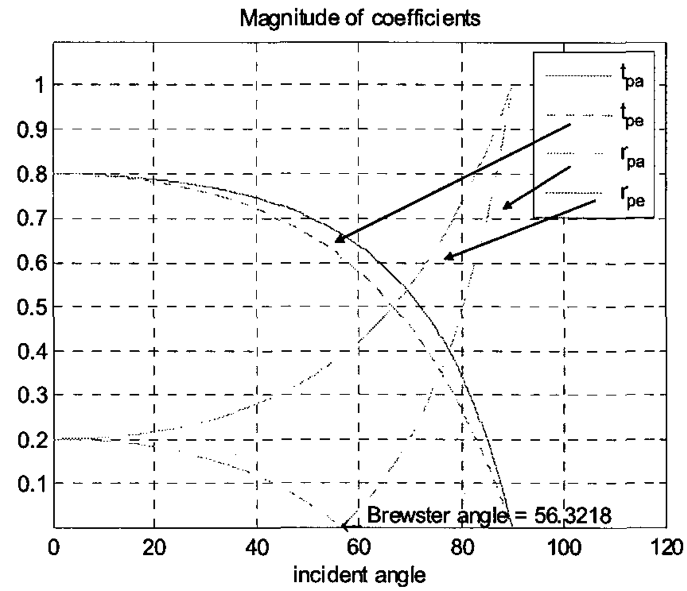
For nonmagnetic materials, i.e., $\mu_1 = \mu_2$,

$$\theta_{p\parallel} = \sin^{-1}\sqrt{\frac{1}{1+(\epsilon_1/\epsilon_2)}} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \tan^{-1}(\frac{n_2}{n_1}). \qquad (2.2\text{-}51)$$

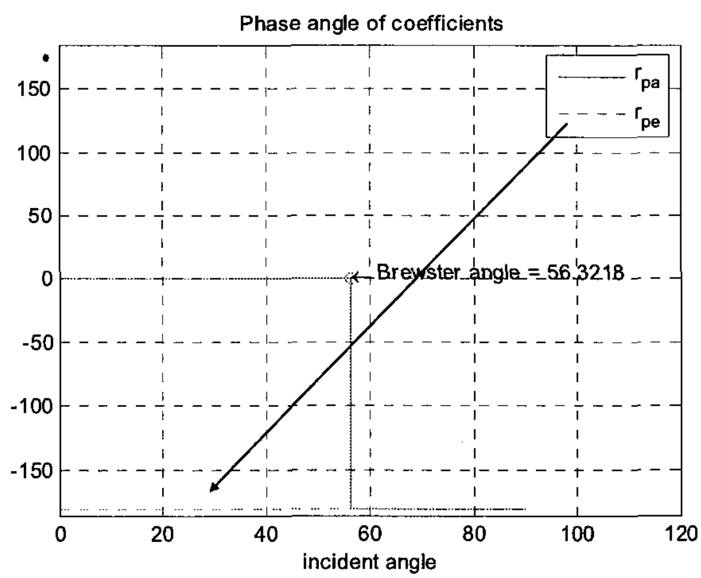
Figure 2.4 plots the reflection and transmission coefficients for incident fields with parallel and perpendicular polarization as a function of the incident angle θ_i for an air-glass interface ($n_1 = 1$, $n_2 = 1.5$, $\mu_1 = \mu_2 = 1$), where we have used $\eta = \sqrt{\mu/\epsilon}$ and $n = \sqrt{\mu_r\epsilon_r}$ to re-express Eqs. (2.2-46) and (2.2-47) in terms of n_1 and n_2. In this case, the coefficients are real as indicated in Fig. 2.4a). Figure 2.4a) can be represented by two figures, Figs. 2.4b) and 2.4c), where in Fig. 2.4c) we see a phase jump from zero degree to -180 degrees at the Brewster angle for the r_{pa} ($r_{\parallel}$) curve.



(a) $r_{\parallel}, t_{\parallel}, r_{\perp}, t_{\perp}$ vs. incident angle. Coefficients are all real in this case.



(b) Magnitude of coefficients vs. incident angle.



(c) Phase angle of coefficients vs. incident angle

Fig. 2.4 Reflection and transmission coefficients for an air-glass interface ($n_1 = 1.0$, $n_2 = 1.5$): t_{pa} and t_{pe} correspond to the cases of transmission coefficient of parallel and perpendicular polarization, respectively. r_{pa} and r_{pe} corresponds to the cases of reflection coefficient of parallel and perpendicular polarization, respectively. These plots are generated using the m-file in Table 2.1.

Reflectivity and transmissivity

It is useful to relate the coefficients of reflection and of transmission to the flow of energy across the interface. Using Eq. (2.2-34), we write the averaged power densities carried by the incident, reflected and transmitted beams, respectively as follows:

$$<\boldsymbol{S}_i> = \frac{|E_{i0}|^2}{2\eta_1}\mathbf{a}_i\ , \quad <\boldsymbol{S}_r> = \frac{|E_{r0}|^2}{2\eta_1}\mathbf{a}_r, \text{ and } <\boldsymbol{S}_t> = \frac{|E_{t0}|^2}{2\eta_2}\mathbf{a}_t\ ,$$

where $\mathbf{a}_i$, $\mathbf{a}_r$ and $\mathbf{a}_t$ are the unit vectors of $\boldsymbol{k}_i$, $\boldsymbol{k}_r$ and $\boldsymbol{k}_t$ as shown in Figs. 2.2 and 2.3. The coefficients of reflection R and transmission T are defined as the ratios of the average power across the interface. For R, it is given by

$$\mathrm{R} = \frac{|<\boldsymbol{S}_r> \cdot \mathbf{a}_z|}{|<\boldsymbol{S}_i> \cdot \mathbf{a}_z|} = \frac{|E_{r0}|^2 \cos\theta_r}{|E_{i0}|^2 \cos\theta_i}. \tag{2.2-52}$$

Hence, for parallel and perpendicular polarization, we have

$$\mathrm{R}_{\|} = |r_{\|}|^2 \text{ and } \mathrm{R}_{\perp} = |r_{\perp}|^2, \tag{2.2-53}$$

respectively as $\theta_i = \theta_r$. $\mathrm{R}_{\|}$ and $\mathrm{R}_{\perp}$ are also called *reflectivity* or *reflectance* in optics. The coefficient of transmission (also called *transmissivity* or *transmittance* in optics) is

$$\mathrm{T} = \frac{|<\boldsymbol{S}_t> \cdot \mathbf{a}_z|}{|<\boldsymbol{S}_i> \cdot \mathbf{a}_z|} = \frac{|E_{t0}|^2 \eta_1 \cos\theta_t}{|E_{i0}|^2 \eta_2 \cos\theta_i}. \tag{2.2-54}$$

Hence, for parallel and perpendicular polarization, we have

$$\mathrm{T}_{\|} = \frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}|t_{\|}|^2 \text{ and } \mathrm{T}_{\perp} = \frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}|t_{\perp}|^2. \tag{2.2-55}$$

Note that conservation of power requires that

$$\mathrm{R}_{\perp} + \mathrm{T}_{\perp} = 1 \quad \text{and} \quad \mathrm{R}_{\|} + \mathrm{T}_{\|} = 1. \tag{2.2-56}$$

As a practical example, for normal incidence $(\theta_i = \theta_t = 0)$ from air $(n_1 = 1)$ to glass $(n_2 = 1.5)$,

$$R_{\parallel} = R_{\perp} = |r_{\parallel}|^2 = |r_{\perp}|^2 = [(n_1 - n_2)/(n_1 + n_2)]^2 = 0.04,$$

and $T_{\perp} = T_{\parallel} = 0.96$. Hence, about 4% of the light is reflected and 96% is transmitted into glass.

Total internal reflection

Recall from Section 1.2 that for $n_1 > n_2$, any light ray incident at an angle greater than the critical angle, $\phi_c = \sin^{-1}(n_2/n_1)$, experiences total internal reflection. What is the picture in terms of wave theory? It turns out Fresnel's equations are all applicable to total reflection if we disregard the fact that $\sin\phi_t > 1$ and for $\phi_i > \phi_c$, we set

$$\begin{aligned}\cos\phi_t &= -(1 - \sin^2\phi_t)^{\frac{1}{2}} \\ &= -[1 - (\frac{n_1}{n_2})^2\sin^2\phi_i]^{\frac{1}{2}} \\ &= \pm j[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}. \end{aligned} \qquad (2.2\text{-}57)$$

Hence, from Eq. (2.2-41). We have, for reflected field,

$$\boldsymbol{E}_r = \boldsymbol{E}_{r0}\exp[-j(\boldsymbol{k}_r \cdot \boldsymbol{R})] = \boldsymbol{E}_{r0}\exp[-j(k_r\sin\theta_r x - k_r\cos\theta_r z)],$$

and the transmitted field is

$$\begin{aligned}\boldsymbol{E}_t &= \boldsymbol{E}_{t0}\exp[-j(\boldsymbol{k}_t \cdot \boldsymbol{R})] = \boldsymbol{E}_{t0}\exp[-j(k_t\sin\theta_t x + k_t\cos\theta_t z)] \\ &= \boldsymbol{E}_{t0}\exp[-jk_t(\frac{n_1}{n_2})\sin\phi_i x]\exp\{-k_t[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}z\},\end{aligned}$$

where we have only retained the real exponential in z with a negative argument to prevent nonphysical solutions. We see that the transmitted field is propagating along the x-direction, with an exponentially decaying amplitude in the z-direction. Such a wave is called an *evanescent wave*.

Taking the case that the incident wave is polarized with its electrical field perpendicular to the plane of incidence [see Fig. 2.3], from Fresnel's equations (2.2-47) and the fact that $\cos\phi_t$ is now an imaginary quantity, $r_{\perp}$ and $t_{\perp}$ become complex and we find

$$r_{\perp} = |r_{\perp}|\exp(j\alpha) = \exp(j\alpha) \tag{2.2-58}$$

and

$$t_{\perp} = |t_{\perp}|\exp(j\alpha/2) = \frac{2\cos\phi_i}{\sqrt{1-(n_2/n_1)^2}}\exp(j\alpha/2), \tag{2.2-59}$$

where

$$\alpha = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\phi_i - (n_2/n_1)^2}}{\cos\phi_i}\right)$$

is the phase angle of the reflection coefficient. We can now write, assuming $\boldsymbol{E}_{i0} = E_{i0}\mathbf{a}_y$,

$$\begin{aligned}\boldsymbol{E}_r &= \boldsymbol{E}_{r0}\exp[-j\,(\boldsymbol{k}_r\cdot\boldsymbol{R})] \\ &= E_{i0}\mathbf{a}_y\exp(j\alpha)\exp[-j\,(\boldsymbol{k}_r\cdot\boldsymbol{R})]\end{aligned} \tag{2.2-60}$$

and

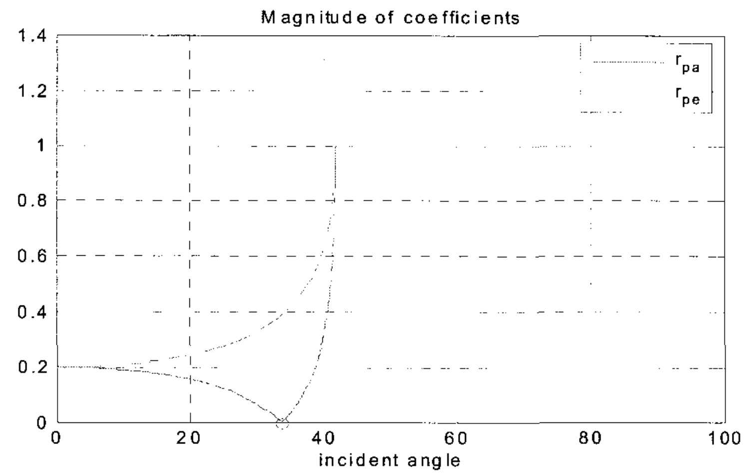
$$\begin{aligned}\boldsymbol{E}_t &= \boldsymbol{E}_{t0}\exp[-j\,(\boldsymbol{k}_t\cdot\boldsymbol{R})] \\ &= E_{i0}\mathbf{a}_y|t_{\perp}|\exp(j\alpha)\exp[-jk_t(\frac{n_1}{n_2})\sin\phi_i x] \\ &\quad\times\exp\{-k_t[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}z\}.\end{aligned} \tag{2.2-61}$$

We notice that the amplitude of the reflected wave is equal to that of the incidence and hence the energy is totally reflected [see Fig. 2.5a)]. However, there is a phase change upon reflection, which varies from 0^0 at the critical angle to 180^0 at grazing incidence as illustrated in Fig. 2.5 b). The results of Fig. 2.5 are plotted using the m-file shown in Table 2.2. Now, the corresponding magnetic field for the transmitted field is, with reference to Fig. 2.3 and using Eq. (2.2-57),

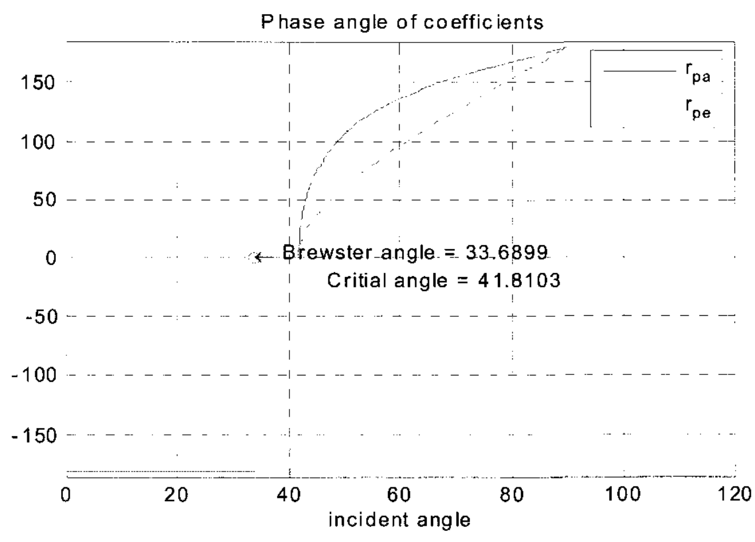
$$\begin{aligned}\boldsymbol{H}_t &= \boldsymbol{H}_{t0}\exp[-j(\boldsymbol{k}_t\cdot\boldsymbol{R})] \\ &= (-\cos\theta_t\mathbf{a}_x + \sin\theta_t\mathbf{a}_z)\,\frac{E_{i0}|t_{\perp}|}{\eta_2}\exp(j\alpha)\exp[-jk_t(\frac{n_1}{n_2})\sin\phi_i x] \\ &\quad\times\exp\{-k_t[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}z\}.\end{aligned}$$

and it becomes the following equation when we use Eq. (2.2-57):

$$\boldsymbol{H}_t = \left\{ j[(\frac{n_1}{n_2})^2 \sin^2\phi_i - 1]^{\frac{1}{2}} \mathbf{a}_x + (\frac{n_1}{n_2}) \sin\phi_i \mathbf{a}_z \right\} \frac{E_{i0}|t_\perp|}{\eta_2} \exp(j\alpha)$$
$$\times \exp[-jk_t(\frac{n_1}{n_2}) \sin\phi_i x] \exp\{-k_t[(\frac{n_1}{n_2})^2 \sin^2\phi_i - 1]^{\frac{1}{2}} z\}. \tag{2.2-62}$$



(a) Magnitude of reflection coefficients vs. incident angle



(b) Phase angle of reflection coefficients vs. incident angle

Fig. 2.5 Reflection coefficients for an glass-air interface ($n_1 = 1.5$, $n_2 = 1$): r_{pa} and r_{pe} corresponds to the cases of reflection coefficient of parallel and perpendicular polarization, respectively.

The time-averaged power density $< \boldsymbol{S}_t >$ of the transmitted field is then

$<\boldsymbol{S}_t> \propto Re[\boldsymbol{E}_t \times \boldsymbol{H}_t^*].$

$$= Re\Big\{[-j[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}\mathbf{a}_z + (\frac{n_1}{n_2})\sin\phi_i\mathbf{a}_x]$$

$$\times \frac{E_{i0}^2|t_\perp|^2}{\eta_2}\exp\{-2k_t[(\frac{n_1}{n_2})^2\sin^2\phi_i - 1]^{\frac{1}{2}}z\}\Big\}. \qquad (2\text{-}2\text{-}63)$$

Note that the transmitted field is obviously not zero ($|t_\perp| \neq 0$), despite the fact that there is no power flowing along the z-direction (as the Poynting vector in the z-direction is imaginary). However, there is power flowing along the interface inside the less denser medium.

If we consider a collection of plane waves (such as a beam), traveling in different directions, to be incident on the interface at angles larger than the critical angle, each plane wave experiences total internal reflection, and the reflection coefficient for each is different. Upon reflection, we can reconstruct the reflected beam by adding the complex amplitudes of each reflected plane wave. The net result is a reflected beam that is laterally shifted along the interface upon reflection. This lateral shift can be interpreted as the energy of the beam entering the less denser medium, traveling along the interface within the less denser medium, and then re-emerging from the less denser medium to the denser medium upon reflection. This lateral shift along the interface of the beam is known as the *Goos-Hänchen shift.*